insensible or almost infinitely small interval of time separating them; for otherwise they would neutralize each other at the moments of break of contact of the original helix connecting the electrodes of the battery.

To Dr. Callan we must all feel deeply indebted for the amount of labour, care, and intelligence he has devoted to chemical electricity, and its extension to the induced electric helix. We must congratulate him, also, on the great success which has attended his improvements and modifications of galvano-electric instruments; which have, by economizing their production, brought them within the means of many experimentalists who, otherwise, could not expect to use or get access to such instruments; and, finally, we may hope that he will continue his exertions, and his liberality in allowing scientific and curious people to see his great instruments in action—a favour which has led me to make this communication, in the hope that it may call more attention to the subject of induced electric action, on the great scale realized by Dr. Callan's *iron* helixes and galvanic batteries.

Mr. John Purser, Jun., M. A., read the following paper:—

On the Application of Corioli's Equations of Relative Movement to the Problem of the Gyroscope.

In treating the problem of determining the apparent* motion of Foucault's gyroscope, different methods have been adopted.† Probably the most satisfactory is that of deducing the equations from the consideration of Corioli's "forces fictives" in relative motion. Corioli has shown that if the co-ordinate axes to which the movement of a system is referred are not fixed, but have a motion of their own in space, we may treat the question in all respects precisely as if these axes were fixed, provided we suppose superadded to the force (P) which acts upon any molecule two others, the first a force (P') equal and opposite to that which would impress on the molecule accelerations equal to those of a point coinciding at the instant with the molecule, but invariably connected with the moving axes—the second force (P'') perpendicular to the relative path of the molecule. Into the value or direction of this last it is unnecessary for the present purpose to enter more particularly.‡

* By apparent motion, here and afterwards, is meant the motion that would be apparent to a spectator on the earth's surface—that is, the motion with respect to co-ordinate axes invariably connected with the earth; by absolute motion, the motion with respect to axes whose direction is fixed in space.

† This is the course taken by M. Quet, in a memoir that appeared on the subject of relative motion, in Liouville's Journal. My apology for reopening the question is, that in that paper the author seems to me to have needlessly complicated the problem by an assumption which, at first sight, appears calculated to simplify it. This will be explained in the sequel.

‡ For the deduction of the expressions for these forces in magnitude and direction, see "Duhamel, Cours de Mecanique," or Corioli's original papers in the "Journal de l'Ecole Polytechnique."

If the connexions of the moving system expressed in relative co-ordinates do not involve the time, we deduce the equation of relative *vis viva* precisely in the same way as that of absolute *vis viva* is obtained when the co-ordinate axes are fixed,—i. e.,

$$\Sigma\,(mv^2) - \Sigma\,(mv_0{}^2) = 2\int_{t_0}^{t}\Sigma\,(mPdp) + 2\int_{t_0}^{t}\Sigma\,(mP'dp'),$$

the $\int\Sigma\,(mP''dp'')$, the work done by the second set of "forces fictives" vanishes, inasmuch as these forces are perpendicular to the displacements of the particles to which they are applied.

When the motion of the moving axes is one of uniform rotation round a fixed line, (P') is evidently a force $(\omega^2 r)$ along the shortest distance from the molecule to the fixed line, and directed outwards from this line, $P'dp' = \omega^2 rdr$,

$$2\int_{t_0}^{t}\Sigma\,(mP'dp') = \omega^2\Sigma m\,(r^2 - r_0{}^2),$$

and the equation of relative *vis viva* assumes the very simple form

$$\Sigma\,(mv^2) - \Sigma\,(mv_0{}^2) = 2\int_{t_0}^{t}\Sigma\,(mPdp) + \omega^2\,(I - I_0),$$

where I and I_0 are the moments of inertia of the moving system round the fixed line at the time (t) and at the origin of time (t_0).

The problem to be solved may be stated as follows:—

A solid of revolution turns round its axés of figure with an angular velocity (n). Its centre of figure being fixed relatively to the earth, and the resultant of the earth's attraction being supposed to pass through this fixed centre, it is required to determine the motion of the axis,

1°. When the axis is restricted to a plane;

2°. When the axis is restricted to a right circular cone;

3°. When the axis is unrestricted.

If we choose for co-ordinate axes three lines at right angles through the centre of the gyroscope moving with the earth, the motion of these axes may evidently be resolved into two—a motion of translation of the origin in a complicated curve in space, and a uniform angular rotation (ω) round an axis* drawn through the origin parallel to the earth's axis. The former evidently does not affect the relative motion of the gyroscope, and may be (as far as the present purpose is concerned) considered as non-existent.

For the complete determination of the motion of a solid body round a fixed point, three equations must be deduced from the dynamical conditions of the problem. In the present instance, the simplest that present themselves are the following:—

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* This axis we shall call, for shortness, the polar line.

I. The component round the axis of figure of the [absolute] angular velocity = Constant = n. This follows directly from Euler's well-known equation for the motion round a principal axis,—

$$C\frac{dr}{dt} = (A - B)\,pq + N.$$

In the present case,

$$A = B \qquad N = 0 \qquad \therefore\ \frac{dr}{dt} = 0 \qquad r = n.$$

Since component of the absolute angular velocity round any line = component of apparent angular velocity + component of angular velocity of the earth, the apparent angular velocity round the axis of figure

$$= n - \omega \cos \theta, \tag{1}$$

where (θ) = angle between axis of figure and polar line.

II. The equation of relative *vis viva*, which in this case assumes the simple form.

$$\Sigma\,(mv^2) - \Sigma\,(mv_0^{\,2}) = \omega^2.\ (I - I_0).* \tag{2}$$

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* It is at this point that my course and my results differ from those of M. Quet. He writes this equation, $\Sigma\,(mv^2) - \Sigma\,(mv_0^2) = 0$. To explain the origin of the discrepancy—instead of choosing our co-ordinate axes passing through the centre of the gyroscope, let us choose them passing through the centre of the earth. The equation of relative *vis viva* would then be

$$\Sigma mv^2 - \Sigma mv_0^2 = 2\int \Sigma m\,Pdp + 2\int \Sigma m\,P'dp'.$$

Where P = force of earth's attraction, P' = centrifugal force due to earth's diurnal rotation. These two forces might be combined for each element into their resultant (R), the force generally understood when we speak of "gravity," and the last member of the equation might be writtten $2\int\Sigma mRdr$. Now, in strict accuracy, neither of these forces P and P' is uniform in magnitude and direction throughout the body of the gyroscope, and, therefore, neither of these integrals vanish. But in seeking to simplify the problem by an assumption sufficiently near the truth, two courses are open to us:—One, that taken by M. Quet to assume the compound force (R) as uniform in magnitude and direction, and that its resultant, accordingly, passes through the centre of figure. He thus gets rid of the second member altogether. The other course, which I have followed here, is to treat the earth's attraction only as uniform, and make no such assumption about the centrifugal force, but to replace $2\int\Sigma mRdr$ by its accurate value, $\omega^2(I - I_0)$. This hypothesis, the uniformity of the earth's attraction, requires only to give it validity that the dimensions of the gyroscope be small compared with the earth; while M. Quet's assumption requires, in addition, that the earth's angular velocity be small compared with that of the gyroscope. Now, it seems more logical, in discussing phenomena arising from the earth's rotation, to include all terms springing from that source. The differential equations so found possess this advantage, that they would not cease to hold good were the earth's angular velocity supposed of co-ordinate magnitude with the gyroscope's. Moreover, applying the equations to the case where the axis of the gyroscope is unconstrained, we obtain on this hypothesis an exact solution; while M. Quet, after an elaborate analysis, has to remain satisfied with an approximation, the simplifying assumption which he made at the beginning precluding him from obtaining a solution in finite terms.

III. The equation of relative moments round the polar line,

$$\Sigma\left(mr^2\frac{d\psi}{dt}\right)-\Sigma\left(mr^2\frac{d\psi}{dt}\right)_0=-\omega\,(I-I_0). \tag{3}$$

Where r = projection of radius vector from the origin to any element on a plane perpendicular to the polar line,

$$\frac{d\psi}{dt}=\text{angular velocity of this projection.}$$

This equation can be very easily proved from the consideration of Corioli's forces; but it is unnecessary to resort to them, for it is evidently but another form of the equation of the conservation of absolute moments round the same line,

$$\Sigma\left(mr^2\frac{d\psi}{dt}\right)-\Sigma\left(mr^2\frac{d\psi}{dt}\right)_0=0,$$

since

$$\text{absolute}\,\frac{d\psi}{dt}=\text{relative}\,\frac{d\psi}{dt}+\omega.$$

Now, let C = moment of inertia round axis of figure,
A = same round any axis perpendicular to this,
$\frac{C}{A}n=m$;

then, since the relative motion of the gyroscope may always be resolved into two, its [apparent] rotation round its own axis, $n-\omega\cos\theta$, and an angular velocity $\frac{ds}{dt}$ round an axis at right angles to its own axis,

$$\text{the relative } vis\ viva=A\left(\frac{ds}{dt}\right)^2+C\,(n-\omega\cos\theta)^2.$$

$$\text{Also } I=C\cos^2\theta+A\sin^2\theta=(C-A)\cos^2\theta+A;$$

$\therefore$ equation (2) assumes the form

$$A\left(\frac{ds}{dt}\right)^2+C\,(n-\omega\cos\theta)^2=\omega^2\,(C-A)\cos^2\theta+\text{Const.}$$

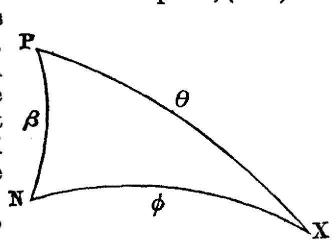
Or,

$$\left(\frac{ds}{dt}\right)^2=2\,m\omega\cos\theta-\omega^2\cos^2\theta+\text{Const.} \tag{4}$$

If the axis is restricted so as to be compelled to trace out a particular curve on the unit sphere, the equation of this curve gives another relation between (s) and (θ), which combined with this determines the motion.

FIRST CASE.—*The Axis is restricted to move in a given Plane.*

Let (P) be the trace of the polar line on the unit sphere, (NX) that of the fixed plane; (X) that of the axis of the gyroscope; or, to define it exactly, of that end of the axis on looking down which the rotation of the gyroscope would appear contrary to the movement of the hands of a watch—that is, would appear in the same direction as the earth's rotation.



Draw the arc PN perpendicular to NX, let $NP = \beta$, $NX = \phi$;

$$\text{then } \cos\theta = \cos\beta\cos\phi, \text{ and } \frac{ds}{dt} = \frac{d\phi}{dt};$$

$\therefore$ by equation (4)

$$\left(\frac{d\phi}{dt}\right)^2 - \left(\frac{d\phi}{dt}\right)_0^2 = 2m\omega\cos\beta(\cos\phi - \cos\phi_0) - \omega^2\cos^2\beta(\cos^2\phi - \cos^2\phi_0). \qquad (5)$$

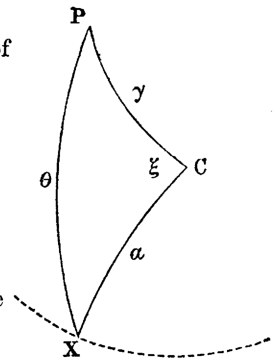
Such is the rigorous differential equation for determining the motion. In its complete form it is unintegrable.

If we confine ourselves to terms of the first order, and suppose the axis of the gyroscope started at relative rest, it becomes

$$\left(\frac{d\phi}{dt}\right)^2 = 2m\omega\cos\beta(\cos\phi - \cos\phi_0).$$

The motion is therefore identical with that of a simple pendulum whose length, $l = \frac{g}{m\omega\cos\beta}$ oscillating about the line (N). When the vibrations are small, the period of a double vibration $T = \frac{2\pi}{\sqrt{m\omega\cos\beta}}$ $= \sqrt{\frac{A\sec\beta}{C}}\,.\,T'$ where T' is a mean proportional between the earth's period of rotation and the gyroscope's.

SECOND CASE.—*The Axis is restricted to a right Circular Cone.*



Let (C) be the trace on the unit-sphere of the axis of the cone (P) and (X) as before.

Let (CX) the angular radius of cone $= a$, $(PC) = \gamma$ angle $PCX = \xi$;

$$\text{then } \frac{ds}{dt} = \sin a \frac{d\xi}{dt},$$

$$\operatorname{Cos} \theta = \cos a \cos \gamma + \sin a \sin \gamma \cos \xi.$$

Equation (4) becomes, on substituting these values, and dividing by $\sin {}^2\gamma$,

$$\left(\frac{d\xi}{dt}\right)^2 - \left(\frac{d\xi}{dt}\right)_0^2 = 2\omega \frac{\sin a}{\sin \gamma} (m - \omega \cos a \cos \gamma) (\cos \xi - \cos \xi_0)$$
$$- \omega^2 \sin {}^2 a (\cos {}^2\xi - \cos {}^2\xi_0) \quad . \quad . \quad . \quad (6)^*$$

Confining ourselves to terms of the first order, and supposing, as before, the axis started at relative rest, we have

$$\left(\frac{d\xi}{dt}\right)^2 = 2 \frac{\sin a}{\sin \gamma} m\omega (\cos \xi - \cos \xi_0).$$

Hence it follows that the axis (X) does not go all round the cone, but vibrates about that edge of the cone which makes the least angle with the polar line, that edge for which $\xi = 0$. The length of the equivalent simple pendulum and the period of a double oscillation, when the vibrations are small, may be found, as in the last case [which is, indeed, included in this as a particular case] to be

$$l = \frac{\sin \gamma}{\sin a} \cdot \frac{g}{m\omega} \qquad T = 2\pi \sqrt{\frac{\sin \gamma}{m\omega \sin a}} = \sqrt{\frac{A \sin \gamma}{C \sin a}} \cdot T'.$$

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* Not long since, Professor Curtis, of Queen's College, Galway, published an interesting paper on this subject. In his investigation of the question he has followed an entirely different method from that here adopted. The origin of the present paper was an endeavour to trace out the cause of the difference between Professor Curtis' results and those arrived at by Professor Price, of Oxford, in the chapter on the gyroscope, in the lately published fourth volume of the Infinitesimal Calculus.

The differential equations (5) and (6) for the motion of the axis, in the last two cases, precisely agree with those given in Professor Curtis' pamphlet, and differ from the corresponding equations in Professor Price's work,—the reason being that the latter follows M. Quet in his assumption, and writes the relative *vis viva* = Const.

THIRD CASE.—*The Axis is unrestricted.*

Denoting as before the polar line and the axis of the gyroscope by P and X, let the angle which the arc (PX) makes with a fixed arc through $(P) = \psi$; the relative angular motion of the gyroscope may be resolved into three rotations:—

$$\begin{cases} n - \omega \cos\theta \text{ round } X; \\ \sin\theta \dfrac{d\psi}{dt} \text{ round an axis in plane } PX \text{ at right angles to } (X); \\ \dfrac{d\theta}{dt} \text{ round an axis perpendicular to plane } (OP). \end{cases}$$

Now, by the equation (3) of relative moments round (O),

$$\sin\theta \,.\, A\sin\theta \frac{d\psi}{dt} + \cos\theta \,.\, C(n - \omega\cos\theta) + (C - A)\,\omega\cos^2\theta = \text{Const.};$$

or, if the axis be started at relative rest,

$$\text{Sin}^2\theta \frac{d\psi}{dt} = -m(\cos\theta - \cos\theta_0) + \omega(\cos^2\theta - \cos^2\theta_0, \tag{7}$$

and by the equation (4) of relative *vis viva*,

$$\text{Sin}^2\theta \left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2 = 2m\omega(\cos\theta - \cos\theta_0) \\ - \omega^2(\cos^2\theta - \cos^2\theta_0) \tag{8}$$

multiplying (7) by (2ω), adding it to (8), and writing ψ' for $\psi + \omega t$, we obtain

$$\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\psi'}{dt}\right)^2 = \omega^2 \sin^2\theta_0; \tag{9}$$

On making the same substitution in (7), it becomes

$$\text{Sin}^2\theta \frac{d\psi'}{dt} = m(\cos\theta_0 - \cos\theta) + \omega\sin^2\theta_0. \tag{10}$$

(ψ') evidently represents the angle the arc (PX) makes with an arc through P retreating with an angular velocity (ω); and the equations (9) and (10) between (θ) (ψ') and (t), are those of the curve described by the axis of the gyroscope with respect to this retreating co-ordinate

arc. A very ready way of integrating these equations is to throw them into the following somewhat different form :—

Let (p) = perpendicular arc let fall from (P) on the great circle tangent to the spherical curve whose running co-ordinates are (θ) and (ψ'); then, by an easy application of Napier's rules for the solution of right-angled spherical triangles,

$$\operatorname{Sin} p = \sin {}^2\theta \, . \, \frac{d\psi'}{ds},$$

$\therefore$ equations (10 and (11) may be written

$$\frac{ds}{dt} = \text{const} = \omega \sin \theta_0, \tag{11}$$

$$\operatorname{Sin} p = \frac{m}{\omega \sin \theta_0} (\cos \theta_0 - \cos \theta) + \sin \theta_0. \tag{12}$$

Equation (12) answers to that of a curve in plano in terms of the radius vector and the perpendicular on the tangent. The expression for the radius of spherical curvature corresponding to the well-known formula

$$R = \frac{rdr}{dp}$$

is

$$\operatorname{Cot} R = - \frac{d \sin p}{d \cos \theta}.$$

[See Graves' translation of Chasles on "Cones and Spherical Conics."]

Applying this expression to the equation of the present curve, we get

$$\operatorname{Cot} R = \frac{m}{\omega \sin \theta_0}, \text{ or } R = \text{const} = \tan^{-1} \frac{\omega \sin \theta_0}{m};$$

$\therefore$ the axis of the gyroscope describes a circular cone of a semi-angle $\tan^{-1} \frac{\omega \sin \theta_0}{m}$, with an angular velocity $\frac{1}{\sin R} \left(\frac{ds}{dt} \right)$

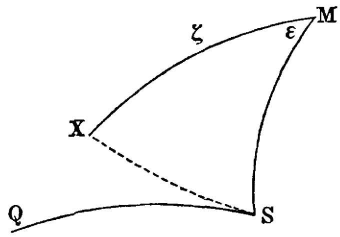
$$= \sqrt{m^2 + \omega^2 \sin {}^2\theta_0};$$

while the axis of the cone revolves round the polar line in a direction opposite to the earth's rotation with an angular velocity (ω); in other words, constantly points to the same fixed star.

For completeness, I have thus solved the case where the axis is unconstrained by the same methods as the other two.

A more rapid solution may, however, be obtained by the ordinary equations of [absolute] *vis viva* and absolute moments thus:—

Tracing the absolute motion of the axis in space on the unit-sphere, let (S) be the starting position of the axis, SQ the direction in which from its connexion with the earth, or any other cause, this axis begins to move, (X) any other position of the axis; M, a fixed line in a plane perpendicular to SQ; let $MX = \zeta$, $XMS = \epsilon$, γ = starting angular velocity of (X); then, by equation of absolute *vis viva*,



$$\left(\frac{d\zeta}{dt}\right)^2 + \sin^2\zeta\left(\frac{d\epsilon}{dt}\right)^2 = \gamma^2;$$

and by equation of moments round M,

$$\operatorname{Sin}^2\zeta\left(\frac{d\epsilon}{dt}\right) = m(\cos\zeta_0 - \cos\zeta) + \gamma\sin\zeta_0.$$

Eliminating $\left(\frac{d\epsilon}{dt}\right)$,

$$\sin^2\zeta\left(\frac{d\zeta}{dt}\right)^2 = \gamma^2\sin^2\zeta - \{m(\cos\zeta_0 - \cos\zeta) + \gamma\sin\zeta_0\}^2;$$

or, if M be chosen, so that $\tan MS = \tan\zeta_0 = \frac{\gamma}{m}$

$$\sin^2\zeta\left(\frac{d\zeta}{dt}\right)^2 + (m^2+\gamma^2)(\cos\zeta_0 - \cos\zeta)^2 = 0,$$

which necessitates

$$\left(\frac{d\zeta}{dt}\right) = 0\text{; and } \zeta = \zeta_0 = \tan^{-1}\frac{\gamma}{m}\text{, and } \frac{d\psi}{dt} = \text{const} = \frac{\gamma}{\sin\zeta_0} = \sqrt{m^2+\gamma^2}.$$

If the starting velocity of the axis is solely due to its connexion with the earth before it was set free,

$$\gamma = \omega\sin\theta_0;$$

$$\zeta = \tan^{-1}\frac{\omega\sin\theta_0}{m};$$

$$\frac{d\psi}{dt} = \sqrt{m^2+\omega^2\sin\theta^2_0};$$

or the axis describes a small circular cone, whose semi-angle = $\tan^{-1}\left(\frac{\omega \sin\theta_0}{m}\right)$, with a uniform angular velocity in a period

$$\frac{2\pi}{\sqrt{m^2 + \omega^2 \sin^2\theta_0}}$$

Still more briefly, the same results may be arrived at by the consideration of Poinsot's resultant couple; for it is evident on inspection that the axis M thus chosen is the axis of the resultant couple of all the motion with which the gyroscope is started. Now, the axis and magnitude of the resultant couple remain fixed; therefore M is always this axis, and G its moment,

$$= \sqrt{C^2n^2 + A^2\omega^2 \sin^2\theta_0},$$
$$= A\sqrt{m^2 + \omega^2 \sin^2\theta_0};$$

and since (Cn), the component of the resultant couple round the axis of figure = $G \cos\zeta$, it follows that

$$\cos\zeta = \text{const} = \frac{Cn}{G} = \frac{m}{\sqrt{m^2 + \omega^2 \sin^2\theta_0}}, \text{ or } \tan\zeta = \frac{\omega \sin\theta_0}{m}$$

Again, the component of the resultant couple round an axis in the plane (XM) perpendicular to $(X) = G \sin\zeta = A \sin\zeta \frac{d\epsilon}{dt}$,

$$\therefore \frac{d\epsilon}{dt} = \frac{G}{A} = \sqrt{m^2 + n^2 \sin^2\theta_0}, \text{ as before.}$$

The result in the unrestricted case may be thus recapitulated:— If the axis of the gyroscope could be started in a position of absolute rest, no angular motion being communicated to the axis either by the earth or the experimenter, it must always continue so, pointing to the same fixed star. When it is not so started, but the axis at the moment of detachment has a velocity (γ) in a given plane, it describes a circular cone round a fixed line in space, the semi-angle of the cone being

$$\tan^{-1}\frac{\gamma}{m},$$

and the period of description

$$\frac{2\pi}{\sqrt{m^2 + \gamma^2}}.$$

When this starting velocity (γ) is solely due to its connexion with the earth before detachment, $\gamma = \omega \sin\theta_0$, a quantity generally so small compared to (m), that the minute arch described by the extremity of the axis would appear an absolute point under the most powerful microscope.

It might be supposed that if this infinitesimal nutation were prevented by restricting the axis to a circular cone round the polar line, the axis would still, as before, follow a fixed star. But this is not so: the relative curve described by its extremity is a spherical cycloid, and the initial tendency of the axis, when set free, being to move towards the polar line, it follows that when this motion is prevented, it remains at relative rest.

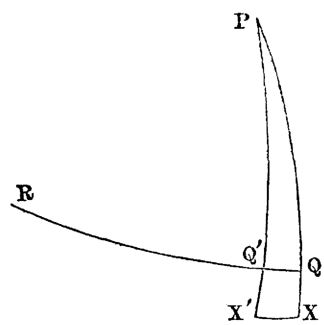
There are one or two points connected with this problem which it may be interesting to examine into.

1°. Supposing the axis of the gyroscope fixed so as to be compelled to move with the earth, what force would it exert to break its bonds?

Let P be the polar line;

XX' two consecutive positions of the axis of the gyroscope;

QQ' the axes of the resultant couple of all the motion the gyroscope has at X and X', then $G = \sqrt{C^2n^2 + A^2\omega^2 \sin^2\theta_0}$, the axis of the couple added by the connexions in the time (dt), which changes the position of G from Q to Q', must lie in the plane QQ' at right angles to Q, the plane of the couple being the plane OQ, let its moment $= Ndt$,



$$\text{then } \frac{Ndt}{G} = \frac{\sin QQ'}{\sin\left(\frac{\pi}{2} - QQ'\right)} = QQ' = XX' \text{ quam proxime,}$$

$$= \omega \sin \theta_0 \, dt,$$

$$\therefore N = G \, . \, \omega \sin \theta_0 = Cn\omega \sin \theta_0 \text{ quam proxime,}$$

that is, the moment of the couple of constraint (N) = that of couple, which, if acting round the axis to stop the spin, would bring the gyroscope to rest in the time $\frac{1}{\omega \sin \theta_0}$, or that of a sidereal day divided by $2\pi \sin \theta_0$.

This will serve as a measure of the friction to be overcome before the apparent motion of the axis could take effect.

2°. In the preceding investigation the resultant of the earth's attraction has been supposed to pass through the centre of the gyroscope, and therefore to exercise no influence on its motion.

In strict accuracy, of course, this is not so, inasmuch as the earth's attraction upon the different parts is neither uniform in magnitude nor direction. The question arises, what is the error induced by supposing it so? Assuming the earth a sphere, it is evident that its attraction has no moment either round the axis of figure, or round the vertical through the centre of the gyroscope.

Choosing this vertical for axis of (z) and the axis of (x) in vertical plane through the axis of the gyroscope, the components of the earth's attraction on any element dm are easily seen to be

$$-\frac{g}{R}x, \quad \frac{-g}{R}y, \quad g+\frac{2g}{R}z,$$

where R = the radius of the earth.

$$\left(\text{Neglecting terms with coefficients } \frac{1}{R^2}.\right)$$

$$\therefore \text{ moment round the axis of } (y) = \Sigma \{ (z\,X - xZ)\,dm \}$$

$$= -\frac{3g}{R}\,\Sigma\,zxdm.$$

To determine this, let $z'x'$ be the co-ordinates with respect to the axis of the gyroscope, and a line at right angles to it in the same vertical plane, the axis of (y) being left unaltered; then

$$\begin{cases} z = z' \cos \nu - x' \sin \nu, \\ x = z' \sin \nu + x' \cos \nu, \end{cases}$$

when ν = inclination of the gyroscope to the vertical;

$$\therefore M = -\frac{3g}{R} \sin \nu \cos \nu \; \Sigma dm\,(z'^2 - x'^2),$$

since $\Sigma dm\,(z'x') = 0,$

or $\dfrac{3g}{R} \sin \nu \cos \nu \,(C - A),$

this moment (M), acting downwards in the vertical plane passing through the axis of the gyroscope, will be the sole effect of the earth's attraction. It will produce terms in the equations with a coefficient

$$\left(\frac{g}{R}\right).$$

These terms will be, of course, inappreciable when compared with the terms whose coefficient is ($m\omega$); but they will be far greater than the terms which have (ω^2) as a factor. We cannot, therefore, in these equations make (m) equal cypher, and assume that the result will represent what happens when the gyroscope is started without any motion round its axis.

All such conclusions would be based on the imaginary hypothesis of the equality of the earth's attraction at different points of the gyroscope.

That the inequality of attraction would materially affect the result when the velocity of the spin is of the same order as (ω) may be shown as follows:—Supposing the gyroscope placed in its frame without spin,

and leaving out of consideration the rotation of the earth, its motion would be that of an oscillation in a vertical plane, determined by the equation

$$A \frac{d\nu^2}{dt^2} = \frac{3g}{R} (C - A) \sin 2\nu.$$

When the starting position of the axis is but slightly inclined to the vertical, and the oscillations are small,

$$\text{the period of vibration} = \sqrt{\frac{R}{6g}} \cdot \sqrt{\frac{A}{C - A}}$$

$$= \sqrt{\frac{A}{C - A}}\ 5\tfrac{1}{2} \text{ minutes, nearly,}$$

a motion far more rapid than in this case (i. e., when the gyroscope is placed in its frame without spin) could arise from the earth's rotation.

3°. In the preceding analysis the problem discussed has had a purely theoretical significance, the rings which realize the conditions proposed being left out of consideration. How will their inertia modify the results? In the first two cases treated there is no difficulty in including them in the moving system. Suppose in Case I. the axis confined to a plane by rendering immoveable the outer ring; let C_1 A_1 be the moments of inertia of the inner ring round an axis perpendicular to its plane, and an axis in its plane; applying the equation of relative *vis viva* to the whole moving system, the equation which replaces (5) will be

$$\left(\frac{d\phi}{dt}\right)^2 - \left(\frac{d\phi}{dt}\right)_0^2 = 2 \frac{C}{A + A_1} \cos \beta \,.\, \omega\, (\cos \phi - \cos \phi_0)$$
$$- \frac{A + C_1 - A_1}{A + A_1}\, \omega^2 \cos^2\beta\, (\cos^2\phi - \cos^2\phi_0)\,.$$

If we compare this with equation (5), it is evident that, omitting terms in (ω^2), the only change to be made in the solution of that case is to suppose (m) to represent

$$\left(\frac{C}{A + A_1} n\right), \text{ instead of } \left(\frac{C}{A} n\right), \text{ as before.}$$

Again, the axis may be restricted to a right circular cone (as in Case II.), by connecting together the two rings, their planes being set making with each other an angle (α) equal to the angular radius of the required cone, and leaving the exterior ring free to revolve round one of its own diameters. Neglecting terms in (ω^2), the results already obtained hold, supposing (m) now to stand for

$$\frac{Cn \sin^2 a}{A \sin^2 a + A_2 + A_1 \cos^2 a + C_1 \sin^2 a}.$$

Lastly, in "the unrestricted case," where both rings must be left free to move, let the line round which the outer revolves be placed parallel to the earth's axis. Including the rings in moving system in this case, and applying as before the equations of relative *vis viva* and relative moments, I have reduced the determination of the motion of the axis to the following pair of equations:—

$$\left\{\begin{aligned} &(A+A_1)\left(\frac{d\vartheta}{dt}\right)^2+\frac{\{Cn\,(\cos\theta_0-\cos\theta)+\omega H_0\}^2}{H}=\omega^2 H_0 \qquad (15)\\ &\frac{d\psi}{dt}+\omega=\frac{Cn\,(\cos\theta_0-\cos\theta)+\omega H_0}{H} \qquad (16)\end{aligned}\right.$$

where $H=A\sin^2\theta+A_1\cos^2\theta+C_1\sin^2\theta+A_2$.

It will be at once seen that an exact solution to correspond with a solution of this case, when the rings are not included, is not to be hoped for. It may, however, be readily shown that, to a very high degree of approximation, the motion of the axis is still that of a retrograde rotation (ω) round the polar line, combined with an infinitesimal conical nutation; for, equating $\frac{d\theta}{dt}$ to cypher, and neglecting terms in (ω^2), the limiting values of θ will be found to be θ_0 and (θ_0-2p), where

$$p=\frac{\omega H_0}{Cn\sin\theta_0}$$

Assuming $\theta=$ its mean value $[\theta_0-p]+y$, and omitting terms of a higher order than (y), we get on substituting in (15)

$$(A+A_1)\left(\frac{dy}{dt}\right)^2+\frac{C^2n^2\sin\theta_0}{H_0}y^2=\omega^2 H_0,$$

or writing

$$q=\frac{Cn\sin\theta_0}{\sqrt{(A+A_1)\,H_0}}$$

$$\frac{dy}{dt}=-q\sqrt{p^2-y^2} \qquad y=p\cos(qt), \qquad (17)$$

the arbitrary constant vanishing, since $y=p$ when $t=0$.

$$\text{Again, }\frac{d\psi}{dt}+\omega=\frac{Cn\sin\theta_0}{H_0}y=\omega\cos(qt),\ \sin\theta_0\left(\frac{d\psi}{dt}+\omega\right)$$
$$=\left(\frac{dx}{dt}\right),\text{ say }=\omega\sin\theta_0\cos(qt);$$

$$\therefore\ x = p' \sin(qt), \tag{18}$$

$$\text{where } p' = \frac{\omega\sqrt{(A+A_1)}\,H_0}{Cn}.$$

These equations (17) and (18) evidently answer to a nutation of the extremity of the axis, not in a circle, as when the rings are left out of consideration, but in an ellipse whose semi-axes are (p) and (p'), and the period of nutation

$$\frac{2\pi}{q}.$$

MONDAY, MAY 25, 1863.

The VERY REV. CHARLES GRAVES, D. D., President, in the Chair.

The Secretary read the following extract of a letter from F. J. FOOT, Esq., to the Rev. Professor HAUGHTON :—

"*Athlone, May* 13, 1863.

"ON the evening that I read my botanical paper at the Academy, in reply to a question put to me by Dr. Osborne, I stated positively that digitalis grows on the limestone of Burren. Since then I mentioned, at the Natural History Society, of its occurring plentifully in the neighbourhood of Mullingar, and also near this. Now, most of the Floras say of digitalis, that it *does not occur in limestone districts.*

"I find that candour demands of me to modify my statement a little. Quite true that digitalis grows in Burren and in the midland counties; but it always grows on *cherty limestone, or its debris.* I must allow that I never saw either *digitalis* or *heather* growing on *pure unsiliceous* limestone. In Burren there are many very siliceous beds of limestone, and on them, in shady places, digitalis is by no means uncommon. Where it occurs at Mullingar and in this neighbourhood, the beds are what has been called *calp*, i. e. black earthy limestone, with bands of chert and shale.

"In fact, if one meets digitalis in a limestone district, they may feel pretty certain that they are on, or very near to, the black calpy limestone."

The Rev. Samuel Haughton, M. D., read a paper "On the Chemical and Mineral Composition of the Granites of Donegal."